Growth rate of small-scale dynamo at low magnetic Prandtl numbers

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Abstract. In this study we discuss two key issues related to a small-scale dynamo instability at low magnetic Prandtl numbers and large magnetic Reynolds numbers, namely: (i) the scaling for the growth rate of small-scale dynamo instability in the vicinity of the dynamo threshold; (ii) the existence of the Golitsyn spectrum of magnetic fluctuations in small-scale dynamos. There are two different asymptotics for the small-scale dynamo growth rate: in the vicinity of the threshold of the excitation of the small-scale dynamo instability, $\lambda \propto \ln(\mathrm{Rm}/\mathrm{Rm}^{\mathrm{cr}})$, and when the magnetic Reynolds number is much larger than the threshold of the excitation of the small-scale dynamo instability, $\lambda \propto \mathrm{Rm}^{1/2}$, where $\mathrm{Rm}^{\mathrm{cr}}$ is the small-scale dynamo instability threshold in the magnetic Reynolds number Rm. We demonstrated that the existence of the Golitsyn spectrum of magnetic fluctuations requires a finite correlation time of the random velocity field. On the other hand, the influence of the Golitsyn spectrum on the small-scale dynamo instability is minor. This is the reason why it is so difficult to observe this spectrum in direct numerical simulations for the small-scale dynamo with low magnetic Prandtl numbers.

1. Introduction

Generation of magnetic field by turbulent motions of conducting fluid is a fundamental mechanism of magnetic fields observed in stars, galaxies and planets. There are different kinds of turbulent dynamos: large-scale and small-scale dynamos. The large-scale mean-field dynamo implies that the amplification of magnetic field occurs at scales which are much larger than the maximum scale of the turbulent motion. This kind of dynamo includes: (i) the $\alpha\Omega$ and $\alpha^2\Omega$ dynamos caused by the combined action of the α effect and differential rotation (see, e.g., [1, 2, 3, 4, 5]); (ii) α^2 dynamo in helical turbulence; and (iii) the shear dynamos in non-helical turbulence [6, 7, 8].

On the other hand, generation of magnetic fluctuations occurs at scales which are smaller than the maximum scale of the turbulent motions (see, e.g., reviews [9, 10, 11, 12, 13, 14]). Self-excitation of magnetic fluctuations with a zero mean magnetic field is called a small-scale dynamo. The mechanisms of the small-scale dynamo action are different depending on magnetic Prandtl numbers $\mathrm{Pm} = \nu/\eta$, where ν is the kinematic viscosity of the fluid and η is the magnetic diffusion due to electrical conductivity of the fluid. For large magnetic Prandtl numbers, the self-excitation of magnetic fluctuations is caused by the random stretching of the magnetic field by the smooth velocity fluctuations (see, e.g., [15, 16, 17, 9, 10, 18, 19, 20, 21]). This type of dynamo has been comprehensively studied in direct numerical simulations (DNS) of forced turbulence [22, 23, 24, 25] and turbulent convection [26, 27]. The nature of small-scale dynamo for low magnetic Prandtl numbers is different, e.g., it is driven by the inertial-range velocity fluctuations at the resistive scale. The small-scale dynamo at low magnetic Prandtl numbers has been studied analytically (see, e.g., [28, 29, 30, 31, 32, 33, 34]) for a Gaussian white-noise velocity field (so called the Kazantsev-Kraichnan model) and numerically (see, e.g., [14, 35, 36]) in a number of publications. Since the magnetic energy is not conserved, the second moment of magnetic field has anomalous scalings [37, 29].

The small-scale dynamo instability is excited when the magnetic Reynolds number, Rm, is larger than the critical magnetic Reynolds number, $\mathrm{Rm}^{\mathrm{cr}}$. Analytical models based on the Kazantsev-Kraichnan model of a homogeneous, isotropic, non-helical and incompressible velocity field, yield $\mathrm{Rm}^{\mathrm{cr}} \approx 410$ at very low magnetic Prandtl numbers [29]. Compressibility of fluid flow causes strong increase of the critical magnetic Reynolds number at $\mathrm{Pm} \ll 1$ (see [29]). Similar tendency also has been recently demonstrated in analytical study [21] at large Prandtl numbers. Direct numerical simulations of small-scale dynamo in [14, 35, 36] of the Navier-Stokes turbulence show that $\mathrm{Rm}^{\mathrm{cr}}$ is around 200 for small magnetic Prandtl numbers, and it is at three times larger than for the small-scale dynamo at large and moderate Prandtl numbers (see [22, 23, 24]). These DNS results at large and moderate Prandtl numbers are in agreement with different analytical models [10, 18, 32, 21].

The existence of the small-scale dynamo for a large number of turbulent spectra at large Prandtl numbers has been demonstrated in [21]. When $\mathrm{Pm} \sim 1$ the small-scale

dynamo exists even in the regime of very large Mach numbers (see the DNS results in [25]). This study has also shown that, for low Mach numbers (~ 0.1), the ratio of the growth rate of turbulence driven by solenoidal and compressive forcing is about 30. However, for higher Mach numbers (~ 10), this ratio is about 2.

The small-scale dynamo action is different from the turbulent induction effect that cases production of the anisotropic magnetic fluctuations by the tangling of the mean magnetic field by the velocity fluctuations (see, e.g., [38, 39, 40, 41, 42, 43]). This effect cannot be described in terms of the small-scale dynamo instability.

In spite of a number of studies of small-scale dynamo at low magnetic Prandtl numbers, there are some key questions that are subject of discussions in the literature. One of them is related to the scaling for the growth rate λ of small-scale dynamo instability at low magnetic Prandtl numbers in the vicinity of the dynamo threshold. Our analysis performed in this study and even numerical solution of the dynamo equations for a Gaussian white-noise velocity field obtained in [44] imply that there are two different asymptotics for the dynamo instability growth rate: (i) in the vicinity of the threshold of the excitation of the small-scale dynamo instability and (ii) far from the threshold of the small-scale dynamo instability.

Other issue studied here is related to an existence of the Golitsyn spectrum, $k^{-11/3}$, of magnetic fluctuations [38, 39] in the small-scale dynamo with low magnetic Prandtl numbers. This spectrum of magnetic fluctuations has been observed in the laboratory experiments [45, 46], in the large-eddy-simulations [47] and in the direct numerical simulation [14, 36] of the small magnetic Prandtl numbers magnetohydrodynamic (MHD) turbulence. In the present study we discuss conditions for the existence of the Golitsyn spectrum.

The small-scale dynamo mechanism appears to be responsible for the random magnetic fields in the interstellar medium and in galaxy clusters [48, 13, 49, 50, 14]. A number of studies recently pointed out also the relevance of the small-scale dynamo to amplify small seed fields in galaxies and the intergalactic medium (see, e.g., [51, 52, 53, 54, 55]). In particular, DNS study in [54] demonstrated that in the presence of turbulence, weak seed magnetic fields are amplified by the small-scale dynamo during the formation of the first stars. Strong magnetic fields are generated during the birth of the first stars in the universe, potentially modifying the mass distribution of these stars and influencing the subsequent cosmic evolution (see [54]). It was also noted in [53] that the small-scale dynamo is very efficient during the formation of the first stars and galaxies. During gravitational collapse, turbulence is created from accretion shocks, which may act to amplify weak magnetic fields in the protostellar cloud. Such turbulence is sub-sonic in the first star-forming minihalos, and highly supersonic in the first galaxies. It was concluded in [53] that magnetic fields are significantly enhanced before the formation of a protostellar disk, where they may change the fragmentation properties of the gas and the accretion rate.

2. Governing equations

Let us study magnetic fluctuations with a zero mean magnetic field at low magnetic Prandtl numbers. In sections 2-3 we use the Kazantsev-Kraichnan model [28] of the δ-correlated-in-time random velocity field. Using this model allows to get the analytical results for the growth rate of the small-scale dynamo instability. The results remain valid also for the velocity field with a finite correlation time if the second-order correlation functions of the magnetic field vary slowly in comparison to the correlation time of the turbulent velocity field (see, e.g., [10, 56]). The two-point instantaneous correlation function of the magnetic field can be presented in the form

$$\langle b_i(t, \mathbf{x}) b_j(t, \mathbf{y})\rangle = \tilde{W}(t, r)\delta_{ij} + \frac{r\tilde{W}'}{2}\,(\delta_{ij} - r_{ij}), \tag{1}$$

where $\tilde{W}(t, r) = \langle b_r(t, \mathbf{x})\, b_r(t, \mathbf{y})\rangle$ is the longitudinal correlation function, b_r is the component of magnetic field $\mathbf{b}$ in the direction $\mathbf{r} = \mathbf{x} - \mathbf{y}$, $r_{ij} = r_i r_j / r^2$ and $\tilde{W}' = \partial\tilde{W}/\partial r$. This form of the second moment (1) corresponds to the condition $\boldsymbol{\nabla} \cdot \mathbf{b} = 0$ and an assumption of the homogeneous and isotropic magnetic fluctuations. Equation for the function $\tilde{W}(r, t)$ derived in the framework of the Kazantsev-Kraichnan model of a homogeneous, isotropic, non-helical, incompressible and Gaussian white-noise velocity field, reads

$$\frac{\partial\tilde{W}(t, r)}{\partial t} = \frac{1}{m(r)}\tilde{W}'' + \mu(r)\tilde{W}' - \frac{\kappa(r)}{m(r)}\tilde{W}, \tag{2}$$

(see [28, 29]), where

$$\frac{1}{m(r)} = \frac{2}{\mathrm{Rm}} + \frac{2}{3}\,[1 - F(r)], \quad \mu(r) = \frac{4}{mr} + \left(\frac{1}{m}\right)',$$

$$\kappa(r) = \frac{2m}{r}\, f'(r), \quad f(r) = F(r) + rF'/3,$$

and $\mathrm{Rm} = u_0\,\ell_0/\eta \gg 1$ is the magnetic Reynolds number, u_0 is the characteristic turbulent velocity in the integral scale ℓ_0 and $F' = dF(r)/dr$. Hereafter equations are written in dimensionless variables: length and velocity are measured in units of ℓ_0 and u_0. For a homogeneous, isotropic and non-helical (with zero mean helicity) and incompressible turbulent fluid velocity field the correlation function $\langle \tau u_i(\mathbf{x}) u_j(\mathbf{x} + \mathbf{r})\rangle$ is given by

$$\langle \tau u_i(\mathbf{x}) u_j(\mathbf{y})\rangle = \frac{1}{3}\left[F(r)\,\delta_{ij} + \frac{rF'}{2}\,(\delta_{ij} - r_{ij})\right]. \tag{3}$$

The form of the continues function $F(r)$ with different scalings in different ranges of scales is constructed using the following reasoning. The function $F(r) = 1 - \sqrt{\mathrm{Re}}\, r^2$ is in the viscous range of scales, $0 \le r \le \ell_\nu/\ell_0$, while the function $F(r) = 1 - r^{4/3}$ is in the inertial range of scales, $\ell_\nu/\ell_0 < r < 1$. At the boundary of these ranges, $r = \ell_\nu/\ell_0$, these functions coincide, where $\ell_\nu = \ell_0/\mathrm{Re}^{3/4}$ is the viscous scale and ℓ_0 is the integral scale of turbulence.

The solution of Eq. (2) can be obtained using an asymptotic analysis (see, e.g., [10, 29, 30]). This analysis is based on the separation of scales. In particular, the

solutions of Eq. (2) with a variable mass have different regions with different functions $m(r)$, $\mu(r)$ and $\kappa(r)$. Solutions in these different regions and their derivatives can be matched at their boundaries. The results obtained by this asymptotic analysis are presented below.

3. Asymptotic behaviour of the growth rate of magnetic fluctuations

Let us discuss the asymptotic behaviour of the growth rate of magnetic fluctuations with a zero mean for small magnetic Prandtl numbers. We seek for a solution of Eq. (2) for the longitudinal correlation function of the magnetic field in the form: $\tilde{W}(t,r) = \exp(\lambda\, t)\, W(r)$. In the viscous range of scales, $0 \leq r \leq \ell_\nu/\ell_0$, the function $F(r) = 1 - \sqrt{\mathrm{Re}}\, r^2$ and the equation for the function $W(r)$ is given by:

$$r\, W'' + 4\, W' + \frac{10}{3}\, \mathrm{Pr}_m\, \mathrm{Re}^{3/2}\, r\, W = 0, \tag{4}$$

where $W' = dW(r)/dr$. The solution of Eq. (4) is given by

$$W(r) = r^{-3/2} J_{3/2} \left(\sqrt{\frac{10\, \mathrm{Pr}_m}{3}}\, \mathrm{Re}^{3/4}\, r\right) \approx 1 - \frac{\mathrm{Pr}_m\, \mathrm{Re}^{3/2}}{3}\, r^2, \tag{5}$$

(see [29]), where $J_\alpha(y)$ is the Bessel function of the first kind, and we have taken into account that $W(r=0) = 1$.

In the inertial range of scales, $\ell_\nu/\ell_0 < r < 1$, the function $F(r) = 1 - r^{4/3}$ and the equation for the function $W(r)$ is given by:

$$\left(1 + \frac{1}{3}\, \mathrm{Rm}\, r^{4/3}\right) W'' + \frac{4}{r} \left(1 + \frac{4}{9}\, \mathrm{Rm}\, r^{4/3}\right) W' + \mathrm{Rm} \left(\frac{52}{27}\, r^{-2/3} - \frac{\lambda}{2}\right) W = 0, \tag{6}$$

where λ is the growth rate of small-scale dynamo instability. In the range of scales, $\ell_\nu/\ell_0 < r \ll \ell_\eta/\ell_0$ the equation for the function $W(r)$ reads:

$$r^2\, W'' + 4\, r\, W' + \frac{52}{27}\, \mathrm{Rm}\, r^{4/3}\, W = 0, \tag{7}$$

where $\ell_\eta = \ell_0/\mathrm{Rm}^{3/4}$ is the resistive scale. The solution of Eq. (7) is given by

$$W(r) = r^{-3/2} J_{9/4} \left(\sqrt{\frac{13\, \mathrm{Rm}}{3}}\, r^{2/3}\right) \approx 1 - \frac{\mathrm{Rm}}{3}\, r^{4/3}, \tag{8}$$

(see [29]). On the other hand, in the range of scales, $\ell_\eta/\ell_0 \ll r < 1$ the equation for the function $W(r)$ is given by:

$$9\, r^2\, W'' + 48\, r\, W' + \left(52 - \frac{27\, \lambda}{2}\, r^{2/3}\right) W = 0. \tag{9}$$

The solution of Eq. (9) is

$$W(r) = C\, r^{-13/6} K_\alpha \left(\sqrt{\frac{27\, \lambda}{2}}\, r^{1/3}\right), \tag{10}$$

(see [34]), where $K_\alpha(y)$ is the real part of the modified Bessel function (Macdonald function) with $\alpha = (i/2)\sqrt{39}$. This solution is chosen to be finite at large r, with positively defined spectrum, and it has the following asymptotics at scales $r \ll \lambda^{-3/2}$ (see [29]):

$$W(r) = A_1\, r^{-13/6} \cos\left(\sqrt{\frac{13}{12}}\, \ln r + \varphi_0\right), \tag{11}$$

and at scales $\lambda^{-3/2} \ll r \ll 1$ (see [32]):

$$W(r) = A_2\, r^{-7/3} \exp\left(-\sqrt{\frac{27\,\lambda}{2}}\; r^{1/3}\right). \tag{12}$$

Here A_1 and A_2 are the constants which are proportional to the constant C.

In the range of scales $r \gg 1$, the turbulence is absent ($F \to 0$), $1/m = 2/3$, $\mu(r) = 4/mr$ and

$$W(r) = A_3\, r^{-2} \left(\sqrt{\lambda} + r^{-1}\right) \exp(-\lambda r), \tag{13}$$

(see [29]), where A_3 is a constant.

The scaling for the growth rate of small-scale dynamo instability which is far from the threshold, is estimated as inverse resistive time scale:

$$\lambda \sim \frac{u_\eta}{\ell_\eta} \sim \frac{u_0}{\ell_0}\, \mathrm{Rm}^{1/2}, \tag{14}$$

(see [39]), where $u_\eta = (\varepsilon\, \ell_\eta)^{1/3}$ is the characteristic turbulent velocity at the resistive scale, $u_0 = (\varepsilon\, \ell_0)^{1/3}$ and ε is the dissipation rate of turbulent kinetic energy. For the scaling $\lambda \propto \mathrm{Rm}^{1/2}$, the condition $\lambda^{1/2}\, r^{1/3} \gg 1$ implies $r \gg \mathrm{Rm}^{-3/4}$. Note that the matching of the solutions (8), (12) and their derivatives at the boundary of their regions yields the dynamo growth rate (14).

However, the scaling, $\lambda \propto \mathrm{Rm}^{1/2}$, is not valid in the vicinity of the threshold of the dynamo instability. Indeed, in the vicinity of the threshold when $\lambda \to 0$, there is only one range of the solution of Eq. (9), i.e., $\lambda^{1/2}\, r^{1/3} \ll 1$. In this range of scales the solution of Eq. (9) is determined by Eq. (11). The matching of the solutions (8), (11) and (13) and their derivatives at the boundary their regions yields the following growth rate of the small-scale dynamo instability in the vicinity of the threshold:

$$\lambda = \beta\, \ln\left(\frac{\mathrm{Rm}}{\mathrm{Rm}^{\rm cr}}\right), \tag{15}$$

(see Appendix A), where $\beta = 4/3$ is the exponent of the scaling of the correlation function $F(r)$ (i.e., it is the exponent of the turbulent diffusivity scaling). In Fig. 1 we plot the growth rate (15) of small-scale dynamo instability versus $\ln\left(\mathrm{Rm}/\mathrm{Rm}^{\rm cr}\right)$ in the vicinity of the threshold of small-scale dynamo instability. In the same figure we also show the numerical solution (squares) of the dynamo equation (2) performed in [44] (for the Kazantsev-Kraichnan model in inertial range of scales of fluid motions for $\mathrm{Re} \sim 10^8$, and for $\mathrm{Rm} \leq 10^7$), which demonstrates perfect agreement between the scaling (15), shown by solid line, and the numerical solution of the dynamo equation.

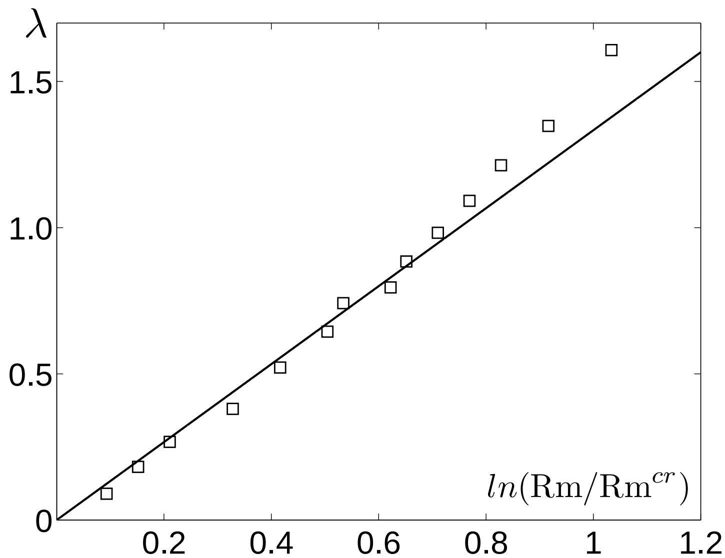

Figure 1. The growth rate of small-scale dynamo instability versus $\ln(\mathrm{Rm}/\mathrm{Rm}^{\mathrm{cr}})$ in the vicinity of the instability threshold: solid line corresponds to the scaling $\lambda = \beta \ln(\mathrm{Rm}/\mathrm{Rm}^{\mathrm{cr}})$ and squares are the results of the numerical solution of the dynamo equation for $W(r)$ for the Kazantsev-Kraichnan model of velocity field with zero kinetic helicity in the inertial range of scales taken from Fig. 1 in [44].

Note that the solution of Eq. (9) determined by Eq. (11), is generally a fast oscillating function at $r \ll 1$. However, for the first mode with the maximum growth rate, the spectrum of the eigenfunction is positively defined (see [10]). Since this solution is only valid in the vicinity of the dynamo threshold, the second and higher modes are not excited. Therefore, the resulting spectrum of the eigenfunctions are always positively defined.

In the present study we discuss only the regime of small magnetic Prandtl numbers. In the case of large magnetic Prandtl numbers and large fluid Reynolds numbers the dynamo growth rate far from the threshold is $\lambda \sim Re^{1/2}$, i.e., it is determined by the Kolmogorov time scale (see, e.g., [39, 20, 21]). On the other hand, for small magnetic Prandtl numbers the dynamo growth rate far from the threshold is determined by the resistive time scale.

4. Magnetic fluctuations with the Golitsyn spectrum

In this Section we study effect of magnetic fluctuations with the Golitsyn spectrum, $k^{-11/3}$, [38, 39] on the small-scale dynamo with low magnetic Prandtl numbers. This spectrum can exist at the scales $\ell_\nu \le r \le \ell_\eta$. Our goal is to determine the longitudinal correlation function $W(r)$ that corresponds to the Golitsyn spectrum. To this end we use the induction equation for the instantaneous magnetic field $\mathbf{H}(t, \mathbf{x})$ in an incompressible

velocity field $\mathbf{v}(t,\mathbf{x})$:

$$\frac{\partial \mathbf{H}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{H} = (\mathbf{H}\cdot\boldsymbol{\nabla})\mathbf{v} + \eta\Delta\mathbf{H}. \tag{16}$$

We seek for the solution of Eq. (16) in the following form:

$$\mathbf{H}(t,\mathbf{x}) = [\mathbf{B}(t) + \mathbf{b}(t,\mathbf{x})]\,\exp(\lambda\, t/2), \tag{17}$$

where $\mathbf{B}(t)$ is the magnetic field in the scales which are much larger than the resistive scale ℓ_η, while $\mathbf{b}(t,\mathbf{x})$ is the magnetic field in the scales which are smaller than ℓ_η. We consider the magnetic dynamo regime, so that the total magnetic field $\mathbf{H}$ grows in time exponentially with the growth rate $\lambda/2$. Now we average Eq. (16) over the ensemble of fluctuations generated in the scales $\ell_\eta \ll \ell \ll \ell_0$, and subtract the obtained averaged equation from Eq. (16). This yields equation for the magnetic field $\mathbf{b}(t,\mathbf{x})$:

$$\frac{\partial \mathbf{b}}{\partial t} = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{u} + (\eta\Delta - \lambda/2)\,\mathbf{b} + \mathbf{b}^N, \tag{18}$$

where $\mathbf{v}(t,\mathbf{x}) = \mathbf{V}(t) + \mathbf{u}(t,\mathbf{x})$, $\mathbf{V}(t)$ is the velocity field in the scales which are much larger than the resistive scale ℓ_η, while $\mathbf{u}(t,\mathbf{x})$ is the velocity field in the scales which are smaller than the resistive scale ℓ_η, and $\mathbf{b}^N = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{b} - \langle\mathbf{u}\times\mathbf{b}\rangle)$ are the nonlinear terms. Equation (18) is written in the frame moving with the velocity $\mathbf{V}(t)$. Using Eq. (18) and the momentum equation for the velocity $\mathbf{u}(t,\mathbf{x})$ we derive equations for the second moments of magnetic field $h_{ij}(\mathbf{k}) = \langle b_i(\mathbf{k})\,b_j(-\mathbf{k})\rangle$ and the cross helicity tensor $g_{ij}(\mathbf{k}) = \langle b_i(\mathbf{k})\,u_j(-\mathbf{k})\rangle$:

$$\frac{\partial h_{ij}(\mathbf{k})}{\partial t} = -\,i(\mathbf{B}\cdot\mathbf{k})\,[g_{ij}(\mathbf{k}) - g_{ji}(-\mathbf{k})] - (\eta k^2 + \lambda/2)\,h_{ij} + h_{ij}^N, \tag{19}$$

$$\frac{\partial g_{ij}(\mathbf{k})}{\partial t} = i(\mathbf{B}\cdot\mathbf{k})\,\tilde f_{ij}(\mathbf{k}) - (\eta k^2 + \lambda/2)\,g_{ij} + g_{ij}^N, \tag{20}$$

where $\tilde f_{ij}(\mathbf{k}) = \langle u_i(\mathbf{k})\,u_j(-\mathbf{k})\rangle$, $h_{ij}^N = \langle b_i^N(\mathbf{k}) b_j(-\mathbf{k})\rangle + \langle b_i(\mathbf{k}) b_j^N(-\mathbf{k})\rangle$ and $g_{ij}^N = -i(\mathbf{k}\cdot\mathbf{B})\,h_{ij}(\mathbf{k}) + \langle b_i^N(\mathbf{k}) u_j(-\mathbf{k})\rangle + \langle b_i(\mathbf{k}) u_j^N(-\mathbf{k})\rangle$. Here u_i^N are the nonlinear terms in the momentum equation. Since we have already taken into account the exponential growth of the total magnetic field $\mathbf{H}$, we can drop the time derivatives in Eqs. (19) and (20), because the characteristic times of the variations of the correlation functions h_{ij} and g_{ij} are much larger than the time λ^{-1}. Since we describe magnetic fluctuations in the spatial scales which are smaller than the resistive scale ℓ_η, we may drop the nonlinear terms h_{ij}^N and g_{ij}^N in Eqs. (19) and (20) in the case of large magnetic Reynolds numbers and low Prandtl numbers, because they are small in these scales. Therefore, the Eqs. (19) and (20) yield:

$$(\eta\, k^2 + \lambda/2)^2\,\langle b_i(\mathbf{k})\,b_j(-\mathbf{k})\rangle = 2(\mathbf{B}\cdot\mathbf{k})^2\,\langle u_i(\mathbf{k})\,u_j(-\mathbf{k})\rangle. \tag{21}$$

Now we introduce the normalized two-point correlation function $w(r)$ of the magnetic field which is defined as follows:

$$w(r) = \frac{1}{\langle B^2\rangle}\,\langle H_m(\mathbf{x})\,H_m(\mathbf{y})\rangle = 1 + \frac{1}{\langle B^2\rangle}\,\langle b_m(\mathbf{x})\,b_m(\mathbf{y})\rangle, \tag{22}$$

where $w' = dw(r)/dr$ and $r = |\mathbf{x}-\mathbf{y}|$. We rewrite Eq. (21) in $\mathbf{r}$ space using the inverse Fourier transformation (i.e., we use the following transformation $ik_i \to \nabla_i$). This yields

the following equation for the normalized two-point correlation function of the magnetic field:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\,\frac{d}{dr} - a^2\right)^2 w(r) = -\frac{2}{3}\,\mathrm{Rm}^2 \left(\frac{d^2 \tilde f(r)}{dr^2} + \frac{2}{r}\frac{d\tilde f(r)}{dr}\right) + 3\,a^4, \qquad (23)$$

where $a = (\lambda\,\mathrm{Rm}/2)^{1/2}$, $\tilde f(r) = \tilde f_{mm}(r)$ and $\tilde f_{ij}(r)$ is the two-point correlation function of the velocity fluctuations written in $\mathbf{r}$ space. Equation (23) is written in dimensionless variables: length and velocity are measured in units of ℓ_0 and u_0, the growth rate of the magnetic fluctuations λ is measured in units of u_0/ℓ_0 and the magnetic field is measured in units of B_0. We also take into account that $\langle(\mathbf{B}\cdot\boldsymbol{\nabla})^2\rangle \tilde f = (1/3)\,(\tilde f'' + 2\tilde f'/r)$.

Solution of Eq. (23) which satisfies the following boundary conditions: $w(r=0) = 3$ and $w'(r=0) = w'''(r=0) = 0$, reads

$$\begin{aligned} w(r) = 3 &- \frac{3\,C_1}{4\,a^3\,r}\,[(ar-1)\,\exp(ar) + (ar+1)\,\exp(-ar)] \\ &+ \frac{55\,\mathrm{Rm}^2}{2\,(3^6)\,a^{11/3}\,r}\Big[(3\,ar-5)\,\exp(ar)\,\gamma(2/3, ar) \\ &+ (3\,ar+5)\,\exp(-ar)\,\gamma(2/3, -ar)\Big], \end{aligned} \qquad (24)$$

where $w' = dw(r)/dr$, C_1 is a free constant and $\gamma(\beta, x) = \beta^{-1}\,x^\beta\,\exp(-x)\,M(1, 1+\beta, x)$ is the incomplete gamma function which is related to the confluent hypergeometric function $M(a, b, x)$. When $ar \ll 1$, Eq. (24) for the two-point correlation function $w(r)$ is given by

$$w(r) = 3 - C_1\,r^2 + \frac{1}{12}\,\mathrm{Rm}^2\,r^{8/3}\left[1 + \frac{9}{238}\,\mathrm{Rm}\,\lambda\,r^2\right]. \qquad (25)$$

The function $w(r)$ is related to the longitudinal correlation function $W(r)$, i.e., $w(r) = 3W(r) + rW'(r)$. Equation (26) rewritten for the longitudinal correlation function $W(r)$, reads

$$W(r) = 1 - \frac{\mathrm{Rm}\,\mathrm{Re}^{1/2}}{3}\,r^2 + \frac{1}{68}\,\mathrm{Rm}^2\,r^{8/3}\left[1 + \frac{9}{322}\,\mathrm{Rm}\,\lambda\,r^2\right], \qquad (26)$$

where $\ell_\nu/\ell_0 \le r \le \ell_\eta/\ell_0$, the constant $C_1 \approx (5/3)\,\mathrm{Rm}\,\mathrm{Re}^{1/2}$ is determined by the matching of functions $W(r)$ determined by Eqs. (5) and (26) at the point $r = \ell_\nu/\ell_0$. The scaling $W(r) \propto \mathrm{Rm}^2\,r^{8/3}$ corresponds to the Golitsyn spectrum $M(k) \propto B_0^2\,\eta^{-2}\,\varepsilon^{2/3}\,k^{-11/3}$ [38], where ε is the rate of dissipation of the turbulent kinetic energy and $M(k) = (2/\pi)\int_0^\infty kr\sin(kr)\,w(r)\,dr$. It follows from the latter equation that the exponent q in the spectrum function $M(k) \propto k^{-q}$ and the exponent p in the scaling $W(r) \propto r^p$ are related as follows $q = p + 1$.

For small Prandtl numbers the constant $\tilde C_1 = \mathrm{Rm}\,\mathrm{Re}^{1/2}/3$ is larger than $\tilde C_2 = \mathrm{Rm}^2/68$, and since $r \ll 1$, the first and second terms in the right hand side of Eq. (26) dominate the behavior of $W(r)$. This estimate implies that the third term in the right hand side of Eq. (26) resembling the Golitsyn spectrum is negligible. This is the reason that the influence of the Golitsyn spectrum on the small-scale dynamo instability is minor. That is why it is so difficult to observe the Golitsyn spectrum in direct numerical simulations (DNS) for the small-scale dynamo with low magnetic Prandtl numbers.

Note that the existence of the Golitsyn spectrum of magnetic fluctuations requires a finite correlation time of the random velocity field, i.e., the solution for the small-scale dynamo with the Golitsyn spectrum does not exist in the framework of the Krachnan-Kazantsev model of the delta-correlated-in-time turbulent velocity field (see Appendix B). Indeed, for the derivation of Eq. (23) we did not used assumption about the delta-correlated-in-time turbulent velocity field. One of the indications of the finite correlation time of random velocity field is already seen in Eq. (23), where the high-order spatial derivatives arise. The Kazantsev-Kraichnan model yields the dynamo equation with spatial derivatives not higher than the second-order spatial derivatives. On the other hand, it is well-known that even small yet finite correlation time of random velocity field causes the appearance of the higher-order spatial derivatives in the dynamo equations (see, e.g., [10, 56]).

The requirement of the finite correlation time of random velocity field for the correct description of the tangling magnetic fluctuations which have the Golitsyn spectrum even follows from the dimensional arguments. Indeed, the main balance in the induction equation for the magnetic fluctuations $(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{u} \sim D\Delta \mathbf{b}$ which yields the Golitsyn spectrum, can be rewritten in the following form: $\langle \mathbf{b}^2 \rangle \sim \tau^2(\ell) \, [\langle \mathbf{u}^2 \rangle]^2 \, \mathbf{B}^2 / D^2$. The latter equation implies the requirement of the finite correlation time of random velocity field for the correct description of the tangling magnetic fluctuations. The similar arguments are also valid for the k^{-1} spectrum of magnetic fluctuations generated by the tangling mechanism at low magnetic Prandtl numbers in the scales $\ell_\eta \ll \ell \ll \ell_0$ (see [40, 41, 42, 43]).

We stress again that both magnetic fields, $\mathbf{B}(t)$ and $\mathbf{b}(t, \mathbf{x})$, are small-scale fields (in scales which are much less than the integral scale ℓ_0 of turbulence). In particular, $\mathbf{B}(t)$ is the magnetic field in the scales $\ell_\eta \ll \ell \ll \ell_0$, while $\mathbf{b}(t, \mathbf{x})$ is the magnetic field in the scales $\ell_\nu \ll \ell \ll \ell_\eta$. These fields belong to the same mode generated by the same small-scale dynamo mechanism. In this section we used two magnetic fields, $\mathbf{B}(t)$ and $\mathbf{b}(t, \mathbf{x})$, to describe interaction of the magnetic fields of different scales by the tangling of the field $\mathbf{B}(t)$ of the velocity fluctuations which produces additional anisotropic magnetic fluctuations with the Golitsyn spectrum. The latter mechanism is the turbulent magnetic induction that is different from the small-scale dynamo.

5. Conclusions

In this study we investigated some key issues of small-scale dynamos in random velocity field with large fluid Reynolds numbers, a zero mean magnetic field and low magnetic Prandtl numbers. Contrary to the claim in [44], there are two different asymptotics for the dynamo growth rate: in the vicinity of the threshold of the excitation of the dynamo instability $[\lambda \propto \ln{(\mathrm{Rm}/\mathrm{Rm}^{\mathrm{cr}})}]$ and far from the dynamo threshold $(\lambda \propto \mathrm{Rm}^{1/2})$. The influence of the Golitsyn spectrum on the small-scale dynamo instability is minor, and this spectrum of magnetic fluctuations requires a finite correlation time of the random velocity field.

We are grateful to Alexander Schekochihin for stimulating discussions who initiated our study of small-scale dynamo with the Golitsyn spectrum. We acknowledge the NORDITA dynamo program of 2011 for providing a stimulating scientific atmosphere.

Appendix A. Growth rate of small-scale dynamo instability in the vicinity of the dynamo threshold

Let us obtain the scaling for the growth rate of the small-scale dynamo instability in the vicinity of the dynamo threshold. In the range of scales, $\ell_\nu/\ell_0 < r \ll \ell_\eta/\ell_0$, the function rW'/W is given by

$$\frac{rW'}{W} = -\frac{4}{9}\,\mathrm{Rm}\, r^{4/3}, \tag{A.1}$$

[see Eq. (8)], while in the range of scales, $\ell_\eta/\ell_0 \ll r < 1$ the function rW'/W is

$$\frac{rW'}{W} = -\frac{13}{6} - \sqrt{\frac{13}{12}}\,\tan\left(\sqrt{\frac{13}{12}}\,\ln r + \varphi_0\right), \tag{A.2}$$

[see Eq. (11)]. On the other hand, in the range of scales $r \gg 1$ the function rW'/W is

$$\frac{rW'}{W} = -2 - \sqrt{\lambda}\, r - \frac{1}{1+\sqrt{\lambda}\, r} \approx -\left(3 + \lambda r^2\right), \tag{A.3}$$

[see Eq. (13)], where we have taken into account that in the vicinity of the dynamo threshold $\lambda \to 0$ and $\lambda^{1/2}\, r \ll 1$.

Matching of the functions rW'/W determined by Eqs. (A.1) and (A.2) at the point $r = \ell_\eta/\ell_0$ yields the following equation:

$$\tan\left(\frac{\sqrt{39}}{8}\,\ln \mathrm{Rm} - \varphi_0\right) = \frac{31}{3\sqrt{39}}. \tag{A.4}$$

This equation determines the function $\varphi_0(\mathrm{Rm})$. Now we define the function $\varphi_0^{\rm cr} = \varphi_0(\mathrm{Rm} = \mathrm{Rm}^{\rm cr})$, where $\mathrm{Rm}^{\rm cr}$ is the threshold for the excitation of the magnetic fluctuations. It follows from this equation that

$$\varphi_0 - \varphi_0^{\rm cr} = \frac{\sqrt{39}}{8}\,\ln\left(\frac{\mathrm{Rm}}{\mathrm{Rm}^{\rm cr}}\right). \tag{A.5}$$

Matching of the functions rW'/W determined by Eqs. (A.2) and (A.3) at the point $r = 1$ yields

$$\tan\varphi_0 = \sqrt{\frac{12}{13}}\,\left(\frac{5}{6} + \lambda\right). \tag{A.6}$$

It follows from this equation that

$$\lambda = \frac{32}{3\sqrt{39}}\,\left(\varphi_0 - \varphi_0^{\rm cr}\right), \tag{A.7}$$

where we have also taken into account that in the vicinity of the dynamo threshold $\lambda \to 0$. Combining Eqs. (A.5) and (A.7), we obtain the following scalinf for the growth rate of small-scale dynamo instability in the vicinity of the dynamo threshold: $\lambda = (4/3)\ln(\mathrm{Rm}/\mathrm{Rm}^{\rm cr})$.

Appendix B. Tangling magnetic fluctuations in the delta-correlated-in-time velocity field

The technique of path integrals for the delta-correlated-in-time velocity field allows us to derive the equation for the second-order correlation function, $h_{ij} = \langle b_i(t,\mathbf{x}) b_j(t,\mathbf{y})\rangle$:

$$\frac{\partial h_{ij}}{\partial t} = [\hat{L}_{ik}(\mathbf{x})\delta_{js} + \hat{L}_{js}(\mathbf{y})\delta_{ik} + \hat{M}_{ijks}]h_{ks} + I_{ij}, \tag{B.1}$$

(see for details [29]), where the turbulent component of magnetic field is $\mathbf{b}(t,\mathbf{x}) = \mathbf{H}(t,\mathbf{x}) - \mathbf{B}(t)$,

$$\hat{L}_{ij} = \frac{1}{3}\left(1 + \frac{3}{\mathrm{Rm}}\right)\delta_{ij}\frac{\partial}{\partial x_p}\frac{\partial}{\partial x_p}, \tag{B.2}$$

$$\frac{1}{2}\hat{M}_{ijks} = \delta_{ik}\delta_{js} f_{mn}\frac{\partial^2}{\partial x_m \partial y_n} - \delta_{ik}\frac{\partial f_{mj}}{\partial y_s}\frac{\partial}{\partial x_m} - \delta_{js}\frac{\partial f_{in}}{\partial x_k}\frac{\partial}{\partial y_n} + \frac{\partial^2 f_{ij}}{\partial x_k \partial y_s}, \tag{B.3}$$

$$I_{ij} = B_k B_s \frac{\partial^2 f_{ij}}{\partial x_k \partial y_s}, \tag{B.4}$$

and $f_{mn} = \langle \tau u_m(\mathbf{x}) u_n(\mathbf{y})\rangle$. Multiplying Eq. (B.1) by $r_i r_j / r^2$ we arrive at the equation for the correlation function $\tilde{W}(r,t)$:

$$\frac{\partial \tilde{W}(t,r)}{\partial t} = \frac{1}{m(r)}\tilde{W}'' + \mu(r)\tilde{W}' - \frac{\kappa(r)}{m(r)}\tilde{W} + I, \tag{B.5}$$

where $I = B^2\,(F'' + 4F'/r)/3$. In the inertial range the source term is $I = 52B^2 r^{-2/3}/27$. This source term yields the following scaling of the correlation function $W(r) \propto r^{4/3}$ in the range of scales, $\ell_\nu/\ell_0 < r \ll \ell_\eta/\ell_0$. This implies that the scaling of magnetic fluctuations caused by the tangling of large-scale magnetic field by the delta-correlated-in-time velocity field coincides with the scaling of turbulent magnetic diffusion $F(r) \propto r^{4/3}$. In Fourier space this corresponds to the $k^{-7/3}$ spectrum of the magnetic fluctuations. This implies that the Golitsyn spectrum, $k^{-11/3}$, of magnetic fluctuations cannot be described in terms of the delta-correlated-in-time velocity field.

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